On the Derivation of the Slutsky Equation in Post War Microeconomics

Alan J. Rogers
Department of Economics
University of Auckland

(a.rogers@auckland.ac.nz)

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Abstract

This paper is concerned with the ways in which the Slutsky equation has been derived in mainstream economics, primarily in the post war period. Also considered are some related results for inverse, or indirect demands, the formal mathematical consideration of which goes back at least as far as Antonelli's ([1886] 1971) work on the integrability problem. The main point is that the traditional approach, which was prevalent until the 1970's, and which is essentially a streamlined version of the calculus-based approach of Pareto ([1927] 1971) and Slutsky ([1915] 1953), can be instructive in ways that complement the modern duality-based approach. The latter has come to monopolise mainstream consumer and producer treatments at the advanced undergraduate and graduate levels and beyond. So far as the Slutsky equation and its analogue for inverse demands are concerned, we argue that the traditional, or classical, approach need not be heavily algebraic, and makes transparent the conditions on preferences beyond the traditional smoothness assumptions required for the derivations of these equations: for direct demands these rule out too much smoothness in indifference curves, and for indirect demands sufficient smoothness is indispensible. The modern approach based on duality arguments yields the same two equations neatly and compactly, but leaves some important ends untied.

1. Introduction

The primary purpose of this paper is to consider the ways in which the Slutsky equation and some related aspects of consumer theory have been derived in treatments of of consumer theory at the advanced undergraduate and graduate levels in the post-war period. These approaches to the theory have become part of neo-classical microeconomic orthodoxy.

The first of two main approaches, which is referred to in this paper as classical, is exemplified by Hicks (1946), Samuelson (1947), and later on by textbooks such as Henderson and Quandt (1958), Allen (1966) and many others, as well as the monograph by Katzner (1970). This approach proceeds by analysing the first order conditions for the problems of maximising a consumer's utility function - assumed sufficiently smooth - subject to a budget constraint, in order to determine the effect of on the maximising choices of small changes in prices and income. The last step, often made use of rather cumbersome arguments involving determinants of certain matrices associated with differentials of first order conditions for the constrained optimisation problem. In fact this approach was the same in essence as that adopted by Pareto ([1927] 1971) and Slutsky ([1915] 1953), and independently by Hicks and Allen (1934a, 1934b).[1]

The prevailing view is that this result, and others, can be much more easily and elegantly obtained by means of the duality between the utility and expenditure functions, with much of the early work on the latter due to Hicks (1956), MacKenzie (1958). For more on this approach, including some historical background, see Diewert (1982). So far as the Slutsky equation itself is concerned, the use of this approach became widespread in the 1970's following the simple derivation in Cook (1972), and subsequently in the influential text by Varian (1978).

The triumph of the duality approach since then seems to have been complete. Modern, comprehensive texts such as Mas-Colell, Whinston and Green (1995) and Jehle and Reny (2001) are unanimous that this is preferable, with the older approach typically not even mentioned as an alternative. A recent exception is Balasko (2011) and in places, Kreps (2013). Of course, one indisputable attraction of duality ideas is that they have many elegant and useful applications such as those suggested early on by Diamond and McFadden (1974), Gorman (1976) and subsequently by many others.

But what seems to have been ignored or forgotten in much of the literature of the last forty years or so, is that it is possible to obtain almost all of the relevant results of modern consumer theory via the classical approach by means of quite elementary mathematics, which certainly should be understandable by those nowadays deemed of capable of studying economics at the graduate level. Morever, the limitations of the classical approach are somewhat more

[1] Chipman and Lenfant (2002) provide considerable historical perspective, and the connection between the work of Pareto and Slutsky is described by Dooley (1983). Schultz (1935) provides an early view and exposition of the work of Slutsky, Pareto, Hicks and Allen, and uses similar mathematics.

apparent than those of the modern alternative, including the conditions imposed on underlying preferences.

This paper is organised as follows. In section 2 we review a classical, pre-duality, way of deriving the Slutsky equation, including explicit consideration of conditions which ensure the differentiability of the compensated and ordinary demand functions. In section 3, some examples are considered which illustrate what can happen when preferences do not satisfy the usual smoothness/curvature properties. In section 4, we consider analogous results, obtained by traditional methods, to those in section 2 for indirect demand functions, and see that the requisite smoothness/curvature properties of preferences differ, but in a way that has a quite clear interpretation. In section 5 we see how direct and inverse demands are related in the "standard" case where the traditional conditions of both section 2 and 4 are satisfied.

In section 6 we turn to a consideration of the duality approach for both direct and indirect demands and the connections between the two that emerge easily under suitable conditions.

Throughout we will be concerned with the limitations of two approaches. This inevitably means a consideration of some technical issues, which turn out to centre on the smoothness and curvature properties of the consumer's preferences. However, the economic interpretations of these and their consequences are not difficult to uncover and understand.

The setting follows the postwar orthodoxy: the consumer chooses a consumption bundles consisting of n goods and represented by an $n \times 1$ vector, x, in the non-negative orthant of n-dimensional Euclidean space (i.e., $\mathbb{R}^n_+$); the consumer's preferences have entirely conventional properties, including monotonicity and strict convexity, and are represented by a continuous utility function, $u(x)$, or simply u, which is everywhere continuous, strictly increasing, and, except where otherwise stated, strictly quasi-concave and smooth.

The consumer's optimisation problem is to choose x to maximise utility subject to a conventional budget constraint: $p'x \leq y$ where y is income (a scalar) and p is a $n \times 1$ vector of positive prices (so $p \in \mathbb{R}^n_{++}$). We also assume away the possibility of binding non-negativity contraints.[2]

It should be noted at the outset that, In part because the analysis in this paper is based on assumed properties of a consumer's preferences, we will not consider the revealed preference approach to consumer demand introduced by Samuelson (1938) and subsequently the subject of considerable development and analysis.[3]

In spite of the fact that much of the paper is concerned with some technicalities and special cases, the discussion is rather informal with preference being given for the most part to notational convenience over rigour.

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[2] This point is important when it comes to differentiability of demand functions, as the example of Kreps (2013, p.275) illustrates.

[3] This theory is formulated in terms of finite differences rather than in terms of the derivatives appearing in the Slutsky equation and elsewhere in the classical and duality based aproaches. Obtaining differential versions of the main revealed preference results is sometimes possible, but we will not pursue that idea here.

2. The Classical Approach

The consumer's utility function u is assumed to possess continuous derivatives up to at least second order; the consumer's optimisation problem is

$$\max_x \quad u(x) \quad \text{subject to} \;\; p'x \leq y.$$

The solution for x to the constrained utility maximisation problem is denoted by $x(p;y)$ (or simply x), and is the vector of ordinary, or Marshallian, demand functions. The first order conditions for an interior maximum are, as usual,

$$\begin{aligned} u_x - \lambda p &= 0 \\ y - p'x &= 0 \end{aligned} \tag{1}$$

where $u_x \equiv \partial u(x)/\partial x$, (the $n \times 1$ vector of marginal utilities) and λ is a Lagrange multiplier. The conditons (1) are sufficient for a local maximum if in addition, u satisfies the second order condition

$$dx'Udx < 0 \;\; \text{for each} \;\; dx \neq 0 \;\; \text{satisfying} \;\; dx'u_x = 0 \tag{2}$$

where U is the Hessian matrix of u. The condition (2) is stated and used by Slutsky ([1915] 1953) and has been widely invoked since.[4] This condition is important in what follows and is equivalent to strong (as opposed to strict) quasi-convexity of u, and implies non-zero Gaussian curvature of the indifference curves.[5]

It is important to note that this condition, i.e, (2), is stronger than required for a solution for x to (1) to yield a local maximum: for example it is possible, in the absence of (2), for u be strictly quasi-concave, so that there is a unique utility maximising x for each p, as the example of Katzner (1968) illustrates.[6] An alternative is to adopt the weaker condition

$$dx'Udx \leq 0 \;\; \text{for each} \;\; dx \;\; \text{satisfying} \;\; dx'u_x = 0 \tag{3}$$

which we will have occasion to refer to.

Note also that nonsingularity of U is, on its own, neither necessary nor sufficient for satisfaction of either (2) or (3): imposing this condition tends to lead, unecessarily, to messy algebraic manipulations which have probably done nothing to make the general approach outlined here digestible.[7]

[4] For example, explicitly by Hicks (1946) and Samuelson (1947), with the latter defining this as a condition for a "regular" relative constrained maximum. See also Allen's (1935) exposition and discussion of Slutsky's paper.

[5] See, for example Barten and Bohm (1982), Malinvaud (1972), Debreu (1972), Mas-Collel (1985).

[6] This and related examples are considered in more detail in the next section. The underlying idea for his case is very simple: the usual second derivative condition is sufficient but not necessary for a local maximum. For example, the function $f(z) = -z^4$ is uniquely maximised at $z = 0$, but there the second derivative vanishes. Neverthless the set $\{y, z | y \leq f(z)\}$ is strictly convex and f is strictly concave. See also Katzner (1970).

[7] For example, Brown and Deaton (1973) assume non-singularity of U, and appear to recommend monotone transformations of u which achieve this in order to obtain explicit formulae for the components of a partitioned inverse of U.

Next totally differentiate the first order conditions (1) to obtain

$$\begin{bmatrix} U & -p \\ -p & 0 \end{bmatrix} \begin{bmatrix} dx \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp \\ -dy + x'dp \end{bmatrix} \tag{4}$$

whenever a solution (to (4)) exists. The bordered Hessian matrix

$$B = \begin{bmatrix} U & u_x \\ u_x' & 0 \end{bmatrix} \tag{5}$$

which, in view of (1), has essentially the same properties, at the optimum, as the matrix on the left hand side of (4). Under (1) and (2) this matrix is easily seen to be non-singular,[8] but need not be under (1) and (3), as Katzner's example shows. When (4) has a unique solution, the ordinary demand functions are everywhere differentiable (in the sense of Dhrymes (1968) and Barton, Kloek and Lempers (1969)),[9] as are also the compensated, or Hicksian, demand functions, as we see below.

For the remainder of this section we will assume that B is non-singular. Then the matrix on the left hand side of (4) is invertible, with inverse denoted by

$$\begin{bmatrix} S & s \\ s' & \sigma \end{bmatrix},$$

so

$$\begin{bmatrix} S & s \\ s' & \sigma \end{bmatrix} \begin{bmatrix} U & -p \\ -p & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}, \tag{6}$$

and we can use (6) obtain explicit expressions for dx in (4). In addition, all the standard results for the Slutsky substitution matrix, which is λS (see below) are readily obtained from (6): symmetry, negative semi-definite, of rank $n-1$, with null space spanned by the vector p. Derivations of these results appear in Appendix A, but they are the same in essence as those in Barton, Kloek and Lempers (1969) and Malinvaud (1972). They are not difficult or algebraically intensive, and they are rigorous under the stated assumptions.

It is important to emphasize that it follows from this derivation that the rank of the Slutsky matrix, λS, cannot be less than $n-1$: this is possible in some settings, but not under the smoothness assumptions made on u in the foregoing.

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[8] Barten and Bohm (1982) show that (1) and (2) are necessary and sufficient for the non-singularity of B under the conditions on u we have imposed (strict quasi-concavity, monotonicity). The term "strong quasi-concavity" is sometimes used for this case. See also Debreu (1972).

[9] The other two possibilities are non-existence of a solution and non-uniqueness, with the latter being, as is usual for systems of linear equations, essentially a local phenomenon. For example, if u is homothetic and U is a matrix of zeros (as can, no solution exists if $dp \neq 0$ unless dp is proportional to p: so only price changes (dp) which are equivalent to income changes (dy) have effects consistent with (4), but in that case the only information (4) contains on dx is that it is such that the budget constraint remains satisfied, i.e., $p'dx = dy - x'dp$.

Note also that under (1) and (2), U can be singular but cannot have rank less than $n-1$, since if it did, B would be singular.[10]

To obtain the Slutsky equation first note from (4) and (6) that

$$dx = \lambda S dp + s(-dy + x'dp) \tag{7}$$

and on setting $dp = 0$, we see that

$$-s = \partial x / \partial y$$

is the vector of income effects. Second, observe that utility is locally constant if $u_x' dx = 0$, i.e., $p'dx = 0$, (using (1)), which is so if and only if $dy = x'dp$, i.e., the change in income is such that the original bundle is still affordable. This notion of income compensation is that employed by Slutsky (1915), but for infinitesimal changes it coincides with the more usual constant utility (or Hicksian) compensation.[11] Putting $dy = x'dp$ in (7) gives the compensated change $dx = \lambda S dp$, which means that λS is the matrix of partial derivatives of the compensated, or Hicksian, demand curves, denoted by $h(p; u)$, or simply h. So

$$\lambda S = \partial h(p; u)/\partial p = \partial h/\partial p,$$

i.e., the Slutsky substitution matrix.[12]

Putting these ideas together and setting $dy = 0$ in (7) yields

$$\frac{\partial x}{\partial p} = \frac{\partial h}{\partial p} - [\frac{\partial x}{\partial y}]x' \tag{8}$$

which is the Slutsky equation. This equation is of interest because of the classical properties of the Slutsky substitution matrix, $\partial h/\partial p = \lambda S$, i.e., symmetry, negative-semidefiniteness, with null space spanned by p, and the economic interpretation of the partial derivatives appearing in it.

All this has been in existence since the late 1960's or earlier, but use was often made use of cumbersome manipulation of determinants in the solution of the linear equations (4). This persisted through the 1970's and in places somewhat later, and is essentially the same approach as Slutsky ([1915] 1953) and Pareto ([1927] 1971): it is the approach adopted by Hicks (1946), Samuelson (1947), and influential textbooks such as Henderson and Quandt (1958) and many others

[10] If U has rank $n-1$, so that $U\phi = 0$ for $\phi \neq 0$, then non-singularity of B requires that $\phi' u_x \neq 0$, or $\phi' p \neq 0$; this is assured in the case where preferences are homothetic and u is chosen to be linearly homogeneous because then $\phi = x$. This case is interesting because $\sigma = 0$ and $s = -x/y$, and these results are related to the connection between the direct and indirect demand functions, as we see below.

[11] Because $dy = p'dx + x'dp$, so $dy = x'dp$ if and only if $p'dx = 0$, i.e., $u_x' dx = 0$ in view of the first order condition (1).

[12] Another way of deriving this result is to consider the problem of minimising $p'x$ by choice of x subject to the utility constraint $u(x) = u^o$. The solution is the vector of compensated demands. Differentiating the first order conditions yields $dx = \lambda S dp$, for these demands, where λ and S are exactly as defined in the text, provided that u^o is equal to the level of utility attained for the original utility maximisation problem.

including, somewhat surprisingly, the unpublished lecture notes by McFadden and Winter (1968) [13].

For later reference we note that the vector of "income effects", $-s$, are, as usual, identical to the effects of a suitable scaling of prices, with income, y, held constant, since if we set $dp = -pd\kappa$ for a scalar κ [14] we obtain $dx = -sx'pd\kappa$ using $Sp = 0$ so that $-s = y\partial x/\partial\kappa$, or

$$-s = \partial x/\partial\kappa,$$

if we adopt the normalisation $y = 1$, noting that $\partial x_i/\partial\kappa$ is non-negative if the i^{th} good is normal. Then (8) takes the form

$$\frac{\partial x}{\partial p} = \frac{\partial h}{\partial p} - [\frac{\partial x}{\partial\kappa}]x'.$$

This formulation is useful for comparisons with inverse demand functions, as we see in section 4.

We conclude this section with a point which is relevant for the following sections, especially section 5. The invertibility of a set of demand functions is distinct from their differentiability. But under the assumptions made in this section, differentiability implies invertibility of the differentials dx, dp. To see this set $dy = 0$ in (7) to obtain

$$dx = [\lambda S + sx']dp. \tag{9}$$

The matrix on the right hand side is nonsingular because $\text{rank}[S] = n - 1$, since $p's = 1$ (recalling $p'S = 0$) or equivalently $x'p = y \neq 0$; then

$$dp = [\lambda S + sx']^{-1}dx \tag{10}$$

which reveals how much p would have to change, with y constant, to induce a given change in the utility maximising choice of x. This idea is explored further in section 4.

A similar idea applies to compensated demands, again under the same assumptions as used in this section: from (7), $dx = \lambda Sdp$ for dp such that $x'dp = dy$ (from the constant utility condition, $p'dx = 0$). Setting $dy = 0$ implies $x'dp = 0$. Now let $\mu \equiv 1/\lambda$; if A is a $n \times n$ matrix, multiplication by μA yields $\mu Adx = ASdp$ and A can be chosen so that $ASdp = dp$ whenever $x'dp = 0$; this is possible because $\text{rank}[S] = n - 1$, and in view of this and $Sp = 0$, A can be chosen (not uniquely) so that $AS = I - px'/y$. Then $\mu Adx = dp$, which reveals how much p would have to change (with

[13] The main thrust of these notes is based on duality ideas, and they are acknowledged by Varian (1978) as an important influence. This is notable because Varian's text was instrumental in establishing the dominance of the duality based approach in graduate level microeconomics. The notes probably also influenced Cook (1972) and Deaton and Muellbauer (1980), since both Cook and Muellbauer studied at Berkeley in the early 1970's, when these notes were in use. Virtually no remnant of the approach taken by Brown and Deaton (1973) survives in Muellbauer and Deaton (1980).

[14] Motivated by considering a small change from p to p/κ.

y constant) to induce a given utility preserving change in x. One choice of A which has some appeal is the reflexive generalised inverse of S because, like S, it is symmetric, negative semidefinite and of rank $n-1$.[15]

3. Examples

Some of the ideas which are important for this paper can be illustrated by an example in which there are just two goods (i.e., $n=2$) and the utility function $u(x)=u(x_1,x_2)$ is strictly quasi-concave and over the region of interest (i.e., open neighbourhoods of those x for which $x_1=x_2$) takes the form

$$u(x_1,x_2)=(x_1+x_2)^\gamma-|x_1-x_2|^\alpha, \qquad x_1,x_2>0 \tag{11}$$

for $\gamma>0$ and $\alpha>1$. Here the parameter α controls the curvature of the indifference curves, with very large values of α corresponding to (locally) almost straight indifference curves, and extreme curvature being associated with values of α close to 1. Note that setting $\gamma=\alpha=1$ in (11) yields

$$u(x_1,x_2)=2\min\{x_1,x_2\}$$

which is the familiar Leontief, or fixed coefficients, utility function with right-angled indifference curves This utility function is non-differentiable, but ordinary and compensated demand are well defined and the Slutsky equation holds: the Slutsky substitution matrix is a matrix of zeros and the ordinary demands are given by

$$x_i=\frac{y}{p_1+p_2}, \qquad i=1,2.$$

This is the textbook example (Kreps (2013), MasCollel, Whinston and Green (1995)) for which the standard results, including differentiablity of demands, hold in the absence of differentability of u.

Hurwicz and Uzawa (1971) present a related but less extreme example where

$$u(x_1,x_2)=3\max\{x_1,x_2\}+2|x_1-x_2| \;+\frac{|x_1-x_2|^{3/2}}{\sqrt{\max\{x_1,x_2\}}}$$

which is similar to (11) for $\gamma=\alpha=1$ but with an additional term: the indifference curves each have a "kink" (at $x_1=x_2$), but are smooth elsewhere and so the ordinary demand functions are not invertible, but are otherwise quite well behaved: in particular the Slutsky matrix is everywhere symmetric and negative semidefinite, and is equal to a matrix of zeros whenever $1/2\le p_1/p_2\le 2$ and elsewhere is of rank $n-1=1$; U is singular for $x_1\ne x_2$ but is not defined at $x_1=x_2$ (with each element of U approaching ∞ as $x_1-x_2\to 0$).

[15] In addition to $SAS=S$ (implied by $AS=I-px'/y$), this choice of A satisfies $ASA=A$ (which ensures that A is negative semidefinite). The two conditions ensure that S and A have the same rank.

Nevertheless, the demand curves possess continuous derivatives everywhere with respect to both prices and income.[16]

Returning to (11), consider the case where $\alpha \in (1, 2)$. Then there is no kink in the indifference curves, both ordinary and compensated demands are continuous, differentiable, and the Slutsky matrix is a matrix of zeros whenever $x_1 = x_2$, but with the usual rank $n-1$ behaviour elsewhere (with elements each converging to zero as $x_1 - x_2 \to 0$). The Slutsky equation as before holds everywhere. The utility function is not twice differentiable where $x_1 = x_2$.

The standard case is obtained by setting $\alpha = 2$ in (11): u is twice continuously differentiable everywhere (in the relevant neighbourhood), U is of constant rank, and S is everywhere of rank $n-1$.[17]

But if $\alpha > 2$, the compensated and ordinary demand functions are not differentiable (at x for which $x_1 = x_2$). The matrix B is singular; the Slutsky equation does not hold because the Slutsky matrix is not defined. Katzner's (1968) example, where

$$u(x_1, x_2) = x_1^3 x_2 + x_1 x_2^3$$

coincides with (11) for $\gamma = \alpha = 4$.

The standard results hold for $\alpha = 2$ in (10) but also more generally, when there is neither too little nor too much curvature: in such cases second derivatives exist and do not exhibit a particular type of degeneracy. The exact characterisation of this condition on curvature is non-singularity of the matrix B.

One question which emerges from the foregoing is the following. Suppose demand functions are such that the Slutsky matrix is well defined and everywhere of rank $n-1$ (as well as being symmetric and negative semi-definite), then does this imply smoothness of u? The answer appears to be yes, at least locally, see Blackorby and Diewert (1979). Such conditions on S are equivalent to quite strong restrictions on the curvature properties of the expenditure function, so we return to this question in section 6.

4. Inverse Demands

Here we consider the inverse, or indirect demand functions. Antonelli ([1886] 1971) used these explicitly in his work on the integrability problem, i.e, conditions on a set of inverse demand functions which ensure that they are consistent with maximisation of a utility function with the usual properties. See also Samuleson (1947), Chipman (1971) and Cornes (1992) for more on his work. The key idea in this context is that observed price ratios provide information

[16] The important point from the point of view of Hurwicz and Uzawa is that the integrability conditions hold, i.e., the direct demand functions satisfy conditions which are sufficient to guarantee recoverability of the utility function from direct demands, without requiring that these be uniquely invertible.

[17] For example, if $\gamma = 1$, then U has rank 1 (and is therefore singular), while if $\gamma = 2$, U is non-singular.

on marginal rates of substitution, so if enough is known about the consumer's inverse demands, his preferences can be recovered from them.

Here we confine attention to the properties of the inverse demand functions under assumptions on the utility function made in the last section, and so the analysis has the same traditional utility-based flavour, and the same starting point. We will be particularly concerned with an analogue of the Slutsky equation, and properties of the Antonelli matrix, which is the analogue of the Slutsky matrix for inverse demands. These are somewhat easier to obtain than the results of the last section, and more importantly, their derivation makes use of weaker assumptions. The derivations below are classical in spirit, are quite straightforward, but do not feature in the literature, perhaps because, where inverse demands are considered, it is typically alongside direct demands. (For example in Samuelson (1947), Salvas-Bronsard et al (1977); and also, although in a less classical spirit, Cornes (1996), Jehle and Reny (2001).)

Here we take a consumption bundle x, and obtain prices p and income y which support this in the sense that, given them, x maximises utility. If the first order conditions (1) are satisfied we can write

$$\lambda = \frac{x'u_x}{y}$$

by using $y = p'x$. Then

$$\frac{x'u_x}{y}p = u_x \tag{12}$$

which on division by $x'u_x/y$ gives the inverse demands, $p(x;y)$, explicitly in term of x and y, and is sometimes referred to as the Hotelling-Wold identity. Clearly the utility function needs to be differentiable for (12) to make sense, but if so it is clear that the inverse demands in (12) are unique. The Leontief and Hurwicz-Uzawa examples outlined in the last section show how uniqueness can fail in the absence of differentiability of u. These derivatives (of u) need not themselves be continuous, or differentiable, and if they are not, the inverse demand functions will not generally be differentiable. To see this consider the example (11) of the last section with $\alpha \in (1,\ 2)$: then the inverse demands are not differentiable at x for which $x_1 = x_2$. The intuition here is clear: small movements in x away from such points induce large changes in p because of the sharp curvature of indifference curves at these points. Additional smoothness in u yields further results, including differentiability of the demands in question, so, as for the previous section, we assume that u is twice continuously differentiable.

In what follows it is notationally convenient to let

$$\mu \equiv y/(x'u_x) \quad \text{and} \quad y = 1;$$

the first simply is an abreviation (equivalent to setting $\mu = 1/\lambda$) and the second is a normalisation, adopted because concern is with the effect of changes in x on the vector p/y. Totally differentiate (12) and rearrange slightly to obtain

$$Udx = x'u_x dp + px'Udx + pu_x'dx$$

or,

$$dp = \mu\,[I - px']\,Udx - \mu pu_x'dx \tag{13}$$

which gives dp uniquely for given dx without any assumption beyond sufficient differentiability of u. Next replace u_x' in (13) with p'/μ, add and subtract $\mu[I - px']Uxp'dx$ from both sides of (13), and rearrange to obtain

$$dp = \mu[[I - px']\,U\,[I - xp']\,dx + \mu\{[I - px']\,Uxp' - pp'/\mu]\}dx. \tag{14}$$

Suppose that dx is such that utility is constant, so $p'dx = 0$. Then the second term on the right hand side of (14) vanishes, and

$$dp = \mu Adx \tag{15}$$

where

$$A \equiv [I - px']\,U[I - xp']. \tag{16}$$

In (15), μA is the Antonelli matrix[18], since the change in x leaves utility unchanged and and so the inverse demands in question are compensated inverse demand functions, which we denote by the vector $a(x; v)$; therefore $\mu A = \partial a(x; v)/\partial x$. The Antonelli matrix is the analogue, for inverse demands, of the Slutsky matrix, and has similar properties. It is symmetric, negative semidefinite, and has rank no greater than $n - 1$, and its null space contains x. Symmetry and the rank result are obvious[19] and negative semidefiniteness is a simple consequence of (16) and (3), as shown in Appendix B.

Note in passing that a simple negativity result for compensated demands can also be easily obtained. This is $dx'dp \le 0$, for any utility preserving change in x, since from (13), $p'dx = 0$ implies

$$dx'dp = \mu dx'Udx$$

which is non-positive by (3). This result is implied by negative semi-definiteness of A, but is not sufficient for it. (It does not, for example, imply, as does negative semidefiniteness, that the diagonal elements of A are non-positive.)

To obtain an analogue of the Slutsky equation for inverse demands consider the effect on p of scaling x, by changing from x to x/δ so in (14) set $dx = -xd\delta$ where δ is a positive scalar. Then $Adx = 0$, because $Ax = 0$, and, from (14),

$$dp = \mu\{[I - px']Uxp' - pp'/\mu\}(-xd\delta),$$

[18] This term for the substitution matrix for a set of inverse demand functions was apparently first used by Samuelson (1947).

[19] Symmetry because of the symmetry of U, and $Ax = 0$, simply because $[I - xp']x = 0$ uner the normalisation $y = 1$.

so, using $u_x = p/\mu$ and $p'x = 1$ yields

$$\begin{aligned}\partial p/\partial\delta &= -\mu\{[I - px']Ux - p/\mu\}p'x \\ &= -b\end{aligned}$$

where

$$b \equiv \mu\{[I - px']Ux - p/\mu\}.$$

Using this in (14) yields

$$dp = [\mu A + bp']dx \tag{17}$$

or

$$\frac{\partial p}{\partial x} = \frac{\partial a}{\partial x} - [\frac{\partial p}{\partial\delta}]p' \tag{18}$$

which is an analogue of (8) for inverse demands. Here the Antonelli matrix appears in place of the Slutsky matrix, with prices and quantities interchanged, and the income effect (or equivalently, the effect of scaling prices) is replaced by the effect of scaling quantities.

The scale effect $(\partial p/\partial x)$ in (18) (or b in (17)) gives the effect on p of a scaling in x so it can be expected to take a simple form when preferences are homothetic. This is so, since, in this case, $Ux = 0$, implying $\partial p/\partial\delta = -b = p$. A similar result holds for direct demands under the same condition on preferences. In this case the geometry of the scale effects for the two types of (uncompensated) demands is identical, but that is not the case for non-homothetic preferences, since for direct demands scaling of prices always preserves, up to scale, the vector u_x, and this is not in general accompanied by a scaling of quantites, while for inverse demands, scaling of quantities is not accompanied by scaling of u_x or prices.[20].

The main result is here is that (18) holds, and that the Antonelli matrix is symmetric, negative semidefinite, of rank at most $n-1$, and its null space contains x. These results are obtained from (1), (3) and twice continuous differentiability of u.[21] Obtaining the same results under weaker conditions seems unlikely to be easily achievable, because the vector of inverse demands is essentially the vector of first partial derivatives of the utility function, so differentiablity of the former is almost equivalent to differentiability of the latter.

Notice also that in the foregoing the Antonelli matrix is not required to have maximum rank $(n-1)$, nor is there any counterpart of the requirement in section 2 that B be non-singular. But $\text{rank}[A] = n-1$ is both necessary and sufficient for the invertibility of the differentials dp and dx in the uncompensated and compensated inverse demands, as can be established by using

[20] Notice also that $[I - px']\partial p/\partial\delta = \mu[I - px']Ux$ since $[I - px']p = 0$ under the normalisation $p'x = 1$, and $I - px'$ is an idempotent matrix. This provides an interpretation of the component of the scale effect on prices in the non-homethetic case; and the two components of the vector $\partial p/\partial\delta$, $-\mu[I - px']Ux$ and p, are orthogonal. A similar decomposition can be obtained for direct demands under the conditions of section 2. We can write $\partial x/\partial y = -[I - xp']s - xp's$, with the two components being orthogonal, while $xp's = x$ from (6); in the homothetic case, $s = -x$, the first of the two components vanishes.

[21] If the condition (3) is replaced by (2), then the rank of A is $n-1$, and the results obtained are entirely analogous to those of section 2.

arguments similar to those at the end of section 2. (A derivation appears in Appendix B.) In fact, in this case, the Slutsky equation, and the properties of its constituent parts, can be obtained entirely from the results given above. This result, and its converse, are considered in the next section.

This point is important because it suggests that if A has rank less than $n-1$, the direct demands will not be differentiable and the Slutsky matrix will not be defined, under the differentiability assumptions on u adopted in this section. In fact, the non-singularity of B and the condition $\text{rank}[A] = n-1$ turn out to be equivalent, as shown in Appendix C. So, if $\text{rank}[A] = n-1$, the results of this section must be consistent with those of section 2, and in fact they mirror those results in a natural way, as we see in the next section.

We conclude this section by returning to the example due to Katzner (1968), mentioned in the last section. For x, p such that $x_1 = x_2 > 0$, $p_1 = p_2$ the elements of U take the same non-zero value , and it easy to verify that $A = 0$ in this case, so $\text{rank}[A] = 0 = n-2$; here B is singular (of rank $n = 2$), and so S is not defined. The indifference curves have so little curvature near the points under consideration that small movements in x induce no changes in p. Or, in terms of compensated direct demands, small changes in p result in changes in x so large that the derivatives of these demand curves are not defined.

Similar results hold for similar reasons in (11) with $\alpha > 2$, the simplest being that for which $\gamma = 1$: then $U = 0$, so $A = 0$, and B is singular. In the case where $1 < \alpha < 2$, the elements of U are not defined (are infinite) at x for which $x_1 = x_2$; there the compensated inverse demands are not differentiable, again because small changes in x induce very large changes in p.

5. The Standard Case.

Here we are concerned with the relationship between the differentials of direct and inverse (or indirect) demands. So we confine attention to situations in which both are defined.

The assumptions made initially are those of section 2, including non-singularity of the matrix B in (5). Then S and A of sections 2 and 4 are defined and each have rank $n-1$. At the end of section 2 we saw that the differentials of the direct demands can be inverted, and for consistency we require that the demands so obtained coincide with those of section 4.

For uncompensated demands we require from (9) and (17) that $[\lambda S + sx']$ and $[\mu A + bp']$ be inverses of each other, i.e.,

$$[\lambda S + sx'][\mu A + bp'] = I \tag{19}$$

and this can be seen to hold from the definitions of S, s, A, b in sections 2 and 4.[22]

[22] The matrix on the left hand side of (19) is $SA + \lambda Sbp' + sx'bp'$, using $\mu = 1/\lambda$ and

For compensated demands we require that, for utility preserving changes in x, $dx = \lambda S dp$ and $dp = \mu A dx$ i.e., $dx = \mu\lambda SA dx$, or, in view of $\mu = 1/\lambda$,

$$[I - SA]dx = 0 \text{ whenever } p'dx = 0.$$

This follows from postmultiplication of (19) by dx, or from the definitions of A and S. Similarly, $dp = \mu A dx = AS dp$ whenever $x'dp = 0$ (an implication of the constant utiity condition $p'dx = 0$ and the normalisation on y). So we require that

$$[I - AS]dp \text{ whenever } x'dp = 0$$

and this can also be verified from (19) or from the definitions.

The fact that S and A are reflexive generalised inverses of each other also follows from (19), since postmultiplication by S yields $S = SAS$ and premultiplication by A yields $A = ASA$.

Next we see how the Slutsky equation and its properties can be obtained from knowledge of the Antonelli equation, and its properties, however these have been derived, provided the Antonelli matrix has rank $n-1$, as well as being symmetric and negative semi-definite. Denote it by μA where μ is a scalar, and recall that $Ax = 0$. We can write $dx = \lambda S dp$ where S is chosen to be a reflexive generalised inverse of A and $\lambda \equiv 1/\mu$. This choice of generalised inverse ensures that λS is negative semidefinite, symmetric and has rank $n-1$, like μA. Moreover, the null space of S is spanned by p since $p'dx = \lambda p'Sdp = 0$ whenever $p'dx = 0$, i.e., for all dp such that $x'dp = 0$, and this implies $p'S = 0$, as required. Therefore λS has the properties of the Slutsky matrix.

The differentials for the uncompensated demands can be inverted[23] to yield $dx = [\mu A + bp']^{-1} dp$ and we write

$$[\mu A + bp']^{-1} = \lambda S + \Delta.$$

Here $\Delta dp = 0$ whenever $x'dp = 0$ (which holds when utility and income are constant), which implies that Δ has rank equal to one, and therefore can be written $\Delta = \gamma x'$. It remains to show that γ is equal to $-\partial x/\partial y$. Set $dp = pdk$ where k is a scalar so $dx = [\lambda S + \Delta]pdk = \gamma x'pdk = \gamma dk$ (using $\lambda Sp = 0$ and $x'p = 1$). Then $\gamma = \partial x/\partial k$, or the vector of income effects $-\partial x/\partial y$, because of the equivalence of the effects on the utility maximising choices of x of decreases in income and equiproportionate increases in prices noted at the end of section 2. This yields the Slutsky equation (8).

Parallel arguments can be employed to obtain (17) from the Slutsky equation (8) on the assumption that S in that equation has maximum rank, $n-1$, as well as being symmetric and negative semi-definite, again regardless of how

$x'A = 0$. From the definition of b, $x'b = -x'p = -1$, so $sx'bp' = -sp'$. And, also from the definition of b, $Sb = \mu SUx$, so $\lambda Sbp' = SUxp'$, since $\mu = 1/\lambda$. Now, $SU - sp' = I$ from (6) so $SUxp' = xp' + sp'$. So the matrix in question is $SA + xp'$. From the definition of A, and $Sp = 0$, $SA = SU[I - xp'] = [I + sp'][I - xp'] = I - xp'$, which yields the result.

[23] Because $\mu A + bp'$ can have rank less than $n-1$ only if $bp'x = 0$, i.e., only if $b = 0$, which is impossible under the usual local non-satiation assumption.

these properties have been obtained. The analog of Δ for the present case also has rank one and is of the form $\gamma p'$ where now $\gamma = \partial p/\partial \delta$, the effect of scaling x on p, is obtained from setting $dx = xdk$, so $dp = [\mu A + \gamma p']xdk = \gamma dk$.

The arguments given above rely on the condition that the ranks of A, S be $n-1$ It is possible for either the Slutsky or Antonelli equation to hold in its absence, but apparently not both.

The final point to be made in this section concerns the two matrices

$$\begin{bmatrix} S & x \\ x' & 0 \end{bmatrix}, \qquad \begin{bmatrix} A & p \\ p' & 0 \end{bmatrix}. \tag{20}$$

in which $Sp = 0$ and $Ax = 0$. These are nonsingular if and only if S and A each have rank $n-1$, with null spaces spanned by p and x respectively.[24] And if S, A are reflexive generalised inverses, the bordered matrices in (20) are inverses of each other, given the normalisation $y = 1$.[25] So there is an appealing symmetry between the matrices in (20) and another sense in which A and S are counterparts, as are also p and x. See Salvas-Bronsard et al (1976) and Stern (1986). The matrices in (20) are also associated with optimisation problems which are related to the one we have been concerned with, as we see in the next section.

6. Duality

The duality approach in microeconomics is based on the theory of convex sets, and makes use of the fact that such sets can be described by their tangent planes (or supporting hyperplanes). See Diewert (1982) for a survey of the development of the theory.[26] The sets in the case of consumer theory of most interest are the upper contour sets $\{x : u(x) \geq v\}$ which here are assumed closed and strictly convex. This implies that a utility function, u, which represents these preferences is strictly quasi-concave, as we have assumed in the preceding sections.

The expenditure function is defined by

$$e(p; v) = \min_x p'x \quad \text{subject to} \quad u(x) \geq v$$

[24] This is so since if the first of these is singular, then $Sc + xd = 0$ and $x'c = 0$, with $c \neq 0$ and/or $d \neq 0$; then $c'Sc = 0$, implies that c is proportional to p if S has rank $n-1$. This is impossible in view of $x'c = 0$. Conversely, if rank of S is less than $n-1$, the first bordered matrix in (20) must be singular. The argument for A an the second bordered matrix in (20) follows the same lines.

[25] Let $SA + xp' = I + \dot{C}$. Then $SAS = S + CS$ and $ASA = A + AC$. Also $x = x + Cx$, using $p'x = 1$, and $p' = p' + p'C$. So $CS = 0, Cx = 0$ and $AC = 0$, $p'C = 0$. Hence $C = 0$ by the non-singularity of the bordered matrices in (20).

[26] Early work was done by Hotelling, Wold, Roy. The expenditure function in consumer theory is due to Hicks (1956) and McKenzie (1958), and the analogous construct in producer theory is due to Shephard (1953). A succinct recent perspective on some of the technical aspects is given by Blume (2008).

and is the least cost of attaining utility v at prices p. It is equal to $p'h(p;v)$ where $h(p;v)$ is the expenditure minimising consumption bundle, or vector of compensated demands. The expenditure function is quite easily seen to be concave in p, and to possess the "derivative property" (otherwise known as Shephard's Lemma)

$$\partial e(p;v)/\partial p = h(p;v). \tag{21}$$

So the vector of partial derivatives of the expenditure function is simply the vector of compensated demands. This an example of one of the appeals of the duality approach.[27] Moreover the existence of the derivatives in (20) depends only on the uniqueness of the expenditure minimising x. This is remarkable in view of the identity $e(p;v) = p'h(p;v)$: differentiability of the left hand side might be expected to require differentiability of its components on the right hand side.[28] The ordinary and compensated demands coincide, as usual, when appropriately evaluated, and in particular,

$$x(p, e(p;v)) = h(p;v). \tag{22}$$

Now partially differentiate both sides with respect to p using the chain rule to obtain

$$\partial x/\partial p + [\partial x/\partial y][\partial e/\partial p]' = \partial h/\partial p \tag{23}$$

or, using $x = h = \partial e/\partial p$,

$$\frac{\partial x}{\partial p} = \frac{\partial h}{\partial p} - [\frac{\partial x}{\partial y}]x'$$

which is the Slutsky equation (8), recalling that $\partial h/\partial p$ is the Slutsky matrix, which we will denote by S^o in this section. It remains to establish the symmetry and negative semidefiniteness of $\partial h/\partial p$: the former follows from Young's theorem and the assumed continuous differentiablity of h, while negative semidefiniteness follows from concavity of the expenditure function in p for fixed v.[29] Finally, the rank of S^o is no greater than $n-1$, and p lies in the null space of S^o, because the expenditure minimising x (i.e, $h(p;v)$) is invariant with respect to scaling of p, for given utility, v.

This is very neat and straightforward once the properties of the expenditure function have been understood. But the Slutsky equation is a statement about derivatives of compensated and uncompensated demand functions and the argument given above simply assumes that these derivatives exist. Katzner's (1968)

[27] A similar property (Roy's identity) holds for the indirect utility function and ordinary (uncompensated) demands. It is possible to base a derivation of the Slutsky equation on Roy's identity, but the usual - and simpler - approach is via the expenditure function, as outlined in htis section.

[28] The idea here is that if p^o, p^* are two price vectors and x^o, x^* are the associated (unique) expenditure minimising consumption bundles, $e(p^*;v) = p^{*\prime}x^* \le p^{*\prime}x^o$ and $e(p^o;v) = p^{o\prime}x^o \le p^{o\prime}x^*$, so $(p^* - p^o)'x^* \le e(p^*;v) - e(p^o;v) \le (p^* - p^o)'x^o$; then divide by $||p^o - p^*||$ and take limits to obtain the desired differentiability.

[29] Concavity follows from the inequality $e(p;v) \le p'x$ for any x for which $u(x) \ge v$, with equality for $x = h(p;v)$.

example and that in (11) (for $a > 2$) serve as reminders that this is not assured, even when underlying preferences have quite conventional properties.[30]

And, even if the second partial derivatives of the expenditure function exist and are continuous, so that the above argument concerning the Slutsky matrix is valid, this says nothing about the income effects, and in particular the existence of the derivatives $\partial x / \partial y$.[31]

The examples of Blackorby and Diewert (1979) reveal that an expenditure function which, for fixed v, is concave, homogenous of degree zero and continuously differentiable in p, and strictly increasing in v, may have associated with it a utility function which does not exhibit all of the usual properties of strict quasi-concavity, continuity and local non-satiation. Blackorby and Diewert's first example is

$$e(p; v) = \begin{cases} vp_1 & \text{for} \quad 0 \le v \le 1 \\ p_1 + (v-1)p_2 & \text{for} \quad v > 1 \end{cases}$$

(for $p_1, p_2 > 0$). This is interesting in the present context because ordinary demands are not everywhere differentiable, while the compensated demands do not depend on prices, and so all elements of the Slutsky matrix are zero. The last feature can be expected to arise whenever $e(p; v)$ exhibits little curvature. (A related example is used by Rubinstein (2006) in a Giffen good context.) Another example in this context is Blackorby and Diewert's third example $e(p; v) = v(p_1 + p_2)$, for which the utility function is the fixed coefficients, or Leontief, case mentioned in section 3: $u(x_1, x_2) = \min\{x_1, x_2\}$. Indeed this is an important example of the advantage of the duality/expenditure function approach: substitution effects are obtained here without the non-differentiability of the utility function getting in the way. A less extreme example is a linearly homogeneous version of Katzner's example, adapted to an expenditure function:

$$e(p; v) = v\left[p_1^3 p_2 + p_1 p_2^3\right]^{1/4}.$$

This has the usual properties of an expenditure function, but all second derivatives vanish at p for which $p_1 = p_2$, so that all elements of the Slutsky matrix

[30] The last sentence of Proposition 3 theorem of Diamond and McFadden (1978), says that if u is twice continuously differentiable and B is non-singular, then the Hessian matrix of the expenditure function is twice continuously differentiable and has rank $n-1$. This is is one implication of section 2, and is essentially classical in flavour. A weaker conclusion is provided in the first sentence: the Hessian of $e(p; v)$ exists, is symmetric and negative semidefinite for almost all positive prices whenever a utility maximising vector exists everywhere (and is presumably unique).

[31] A similar point in a different context is made by Hurwicz and Uzawa (1971): integrability of a set of (uncompensated) demand functions is obtained by imposing conditions on the income effects as well on the matrix of subsitution effects. For more on the connection between concepts of smoothness of preferences and smoothness of ordinary and compensated demand functions, see Neilson (1991) and Debreu (1972): for the latter only smoothness/curvature along indifference curves matters, while the former requires smoothness over the indifference map.

are zero at such p. This expenditure function is nevertheless strictly quasi concave in p for fixed v.

Some of the more theoretical work on duality, such as Krishna and Sonnenschein (1990) and Jackson (1986), has emphasized restrictions on classes of expenditure functions and utility functions which are such that there is a one-to-one correspondence between members of each class. This is sometimes known as the "duality problem" because once the correspondence has been established, it is possible to work with either type of function according to convenience, certain in the knowledge of the properties of its partner in the other class, without having to derive those properties explicitly.

It would be convenient to have similar results for the differentiability properties of expenditure and utility functions. Some have been obtained by Blackorby and Diewert (1979): these state that, locally, a "standard" set of second order differentiability properties of the expenditure function implies and is implied by "standard" behaviour of the utility function.

But this does reveal what non-standard behaviour of the expenditure function implies for the utility function, and conversely. For example, if $e(p;v)$ is twice differentiable in p and its Hessian has rank less than $n-1$, $u(x)$ is probably not twice differentiable; conversely, if $u(x)$ is not twice differentiable, the Slutsky matrix may be defined, but if so probably has rank less than $n-1$.

Leaving this point aside, another argument in favour of the duality approach is that it allows easy, parallel derivations for inverse demands by means of the "distance function" whose popularity in consumer theory seems due primarily to Gorman (1976). This is defined by

$$d(x;v) = \max_{\delta}\{u(x/\delta) \geq v\}$$

i.e., for given x and given level of utility v, $d(x;v)$ is the amount by which x must be scaled so that the scaled commodity bundle, x/δ, yields utility v.[32] It is another way of representing preferences, noting in particular that $d(x;v) = 1$ defines the indifference curve corresponding to utility v, (since $d(x;v) = 1$ if and only if $u(x) = v$). Therefore, the expenditure function may be defined in terms of d instead of u as

$$e(p;v) = \min_{x} p'x \quad \text{subject to} \quad d(x;v) \geq 1. \tag{24}$$

Instead of deriving the properties of the distance function from scratch, it is convenient to make use of the relation

$$d(x;v) = \min_{p} p'x \quad \text{subject to} \quad e(p;v) \geq 1 \tag{25}$$

as in Gorman (1976), and also Deaton (1979), Cornes (1995). One way of obtaining (25) is as follows. Take an arbitrary x, p, v with p scaled so that

[32] This defintion is essentially the same as that of the "gauge function" defined by McFadden (1978); see also Rockafellar (1970). Weymark (1980) uses the term "transformation function"; the idea has a long history in the analysis of production technologies.

$e(p;v) = 1$ [33] and choose δ so that $u(x/\delta) = v$. Then $e(p;v) \leq p'x/\delta$, so $d(x;v)e(p;v) \leq p'x$ since $d(x;v) = \delta$ from the definition of d and the choice of δ. Therefore $d(x;v) \leq p'x/e(p;v) = p'x$ which yields (25); noting that the inequality $d(x;v) \leq p'x$ holds as an equality when x/δ is the least cost way of attaining v at prices proportional to p. This last point is important because it is the reason why the solution for p in (25) is the vector of compensated inverse demands, given x and v.

Once (24) and (25) have been established, it is possible to exploit the fact that d has exactly analogous properties as e: the derivative property is $\partial d/\partial x = a(x;v)$, i.e., the vector of compensated inverse demands, and and whenever it is defined, the Hessian of $d(x;v)$, $\partial^2 d(x;u)/\partial x \partial x'$, is symmetric negative semidefinite, of rank at most $n-1$, with x contained in its null space. So we obtain classical results for the Antonelli matrix, since it is precisely this Hessian, and we denote it in this section by A^o.

To obtain an analogue of the Slutsky equation for inverse demands requires defining the uncompensated demands in such a way that the properties of the distance function can be further exploited. The (vector of) uncompensated inverse demands is written $p(x;\delta)$: as before, this is the price vector which supports x/δ as the utility maximising choice when income (y) is equal to one; $p(x;\delta)$ is clearly homogeneous of degree zero in x and δ.

Compensated and uncompensated inverse demands coincide when suitably evaluated: specifically, $p(x;\delta) = a(x;v)$ when δ is such that $u(x/\delta) = v$, or, equivalently,

$$p(x; d(x;v)) = a(x;v).$$

Now differentiate both sides with respect to x, use $\partial d(x;v)/\partial x = a(x;v)$, and evaluate at $\delta = 1$, (so $v = u(x)$) to obtain

$$\frac{\partial p}{\partial x} = \frac{\partial a}{\partial x} - [\frac{\partial p}{\partial \delta}]a'$$

which is (17) of section 4, once a in the second matrix on the right han side is replaced by p, and the scale effect is again given by the vector $\partial p/\partial \delta$, evaluated at $\delta = 1$.

Since the distance function is just another of representing a preference ordering, it shares many of the properties of a utility function used for the same purpose. So differentiability can be an issue: for the Leontief example considered briefly in section 2, the distance function is not differentiable, so the derivative property does not hold. [34] And, the existence of second derivatives can also be problematical. In (11), for example, this fails for the case $\gamma = \alpha \in (1,2)$, so A^o is not defined. This is not surprising in view of the results of section 4, since in this case the utility function is not twice differentiable.

Now we turn to the connection between A^o and S^o. Apply the derivative

[33] This scaling is harmless because $e(p;v)$ is homogeneous of degree one in p.

[34] Which simply reflects the fact that the (normalised) price vector which support an economically relevant consumption bundle is not unique.

property twice to each of $e(p;v)$ and $d(x;v)$ to obtain, as in Deaton (1979),

$$
\begin{aligned}
h(p;v) &= h(\partial d(x;v)/\partial x;v) = \partial e(\partial d(x;v)/\partial x;v)/\partial p && (26)\\
a(x;v) &= a(\partial e(p;v)/\partial p;v) = \partial d(\partial e(p;v)/\partial p;v)/\partial x; && (27)
\end{aligned}
$$

differentiating the first of these with respect to p then yields $S^o = S^o A^o S^o$, and the second with respect to x yields $A^o = A^o S^o A^o$, once we evaluate the derivatives in question at $e(p;u) = y = 1$, (a scale normalisation on prices) and $\delta = 1$ (a scale normalisation on quantities).

This is very neat, but does not give much insight into the economic interpretation of the relationship between S^o and A^o, including the question of the invertibility of the differentials dx and dp discussed in the last section. The derivatives in question obviously must exist for this argument to be valid, but in addition A^o and S^o must have maximum rank, $n-1$, something that is not at all obvious from the argument just given, but which should be clear from the last section, if the elements of these matrices are to be interpreted as partial derivatives of a consumer's compensated direct and inverse demands.

One way of thinking about this last point is to note that the first order conditions for (24)

$$p + \lambda^o \partial d(x;v)/\partial x = 0$$

where λ^o is a Lagrange multiplier; here the derivative property gives $\partial d(x;v)/\partial x = a(x;v)$, so $\lambda^o = 1$, and totally differentiating this and the constraint $d(x;v) = 1$, applying the derivative property again along with $A^o = \partial^2 d(x;v)/\partial x \partial x'$ (and so assuming differentiability of compensated inverse demands) yields

$$\begin{bmatrix} A^o & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} dx \\ d\lambda^o \end{bmatrix} = \begin{bmatrix} dp \\ 0 \end{bmatrix}. \tag{28}$$

If we pursue this approach, differentiability of direct compensated demand functions requires that the matrix on the left hand side of (28) be non-singular, which requires that the rank of A^o be $n-1$. Observe the similarity between (28) and (4) if for the latter we consider compensated demands by setting $dy = x'dp$. This is is not surprising because for compensated direct demands the problem is equivalent to minimising expenditure subject to a utility constraint, and this problem therefore differs from that in (28) only in the way the utility constraint is represented.

If we proceed in the same way with the optimisation problem (25), we obtain the first order condition $x + \mu^o \partial e(p;v)/\partial p = 0$, where μ^o is a Lagrange multiplier, and this and the constraint $e(p;v) = 1$ yields $\mu^o = 1$, and (assuming the differentiability of compensated direct demands),

$$\begin{bmatrix} S^o & x \\ x' & 0 \end{bmatrix} \begin{bmatrix} dp \\ d\mu^o \end{bmatrix} = \begin{bmatrix} dx \\ 0 \end{bmatrix}. \tag{29}$$

In (29), differentiability of indirect demands using this approach requires that S^o have rank $n-1$.

The matrices on the left hand sides of (28) and (29) are inverses of each other, when these exist. Under this condition we can obtain the generalized inverse connection between A^o and S^o given above, and we can obtain (28) and (29) from each other, and so obtain the relationships between dx and dp for the two types of compensated demands. Again this is neat, and the symmetries are appealing, but we have seen from section 4 that approaching indirect demands this way imposes restrictions on consumer preferences that can obscure some fundamental differences between the two types of demand functions.

Conclusion

Katzner (1970), whose book is squarely in the classical tradition, notes that the distinguishing feature of that approach is that it is based on utility maximisation in the presence of certain restrictions, including smoothness of the utility function chosen to represent the consumer's preference ordering. The smoothness assumption usually takes the form of a twice continuous differentiability assumption, and is usually adopted for analytical convenience. Kreps (2013, p.276), for example, has noted that while many economists find it unobjectionable, "it is also without any serious axiomatic basis". The modal approach in the modern literature, on the other hand, is to adopt axioms on a preference ordering such as completeness, continuity, monotonicity, and convexity of upper contour sets, and then proceed to obtain from these properties of the expenditure function. This gives the impression that the older approach makes use of an unnecessary, clumsy, ill-motivated assumption for the sake of some tedious algebraic manipulations which can be avoided by adopting duality arguments.[35] This may be true of some results, but there are others, such as the Slutsky equation for direct demands and its analogue for indirect demands, where quite tidy, self-contained arguments can be constructed along entirely classical lines in such a way that the role of the smoothness/curvature assumptions used is entirely apparent.

The disadvantage of the duality approach to the Slutsky equation is that the latter is a statement about derivatives of compensated and uncompensated demand functions, and duality methods are not conspicuously useful in establishing the differentiablity of these demand functions.[36] The strength of duality

[35] Deaton (1979, p.396) comments: "[m]uch of the power of duality methods comes from their ability to replace mechanical matrix inversion by elementary algebra with economic interpretations". Note, though, that there are no such inversions in section 4. Cornes (1992) also rejoices in the absence of "bordered Hessians" in the duality approach: neverthless, there are occasions where they do arise quite naturally, as we have seen in (28) and (29) in the last section. And the critical curvature condition of section 2 is precisely the non-singularity of the bordered Hessian, B.

[36] For example, Kreps (2013) obtains non-singularity of the bordered B as sufficient for differentiablity of ordinary direct demand functions; this derivation has a decidedly classical flavour (c.f. Dhrymes (1967) , Barten et al. (1969)). The approach of section 2 makes it clear that this issue should be confronted at some stage, and in section 2 this is done prior to a consideration of the Slutsky equation.

methods, as Krishna and Sonnenschein (1990) note, is the use of functions (including the expenditure function and distance function) whose first derivatives are demand curves. But it is less helpful when it comes to the second derivatives of those functions. When this issue is taken seriously (as, for example, in Kreps(2013), Mas-Collel (1985)) the analysis often ends up having a classical look. And one reason for taking that issue seriously is the connection between the Slutsky results (especially symmetry and negative semi-definiteness of the Slutsky substitution matrix) and the integrability problem, i.e., the extent of the restrictions imposed on observable behaviour by the utility maximisation hypothesis.

Appendix A

Properties of S.

Symmetry of S follows from the symmetry of U (a consequence of Young's Theorem and the smoothness assumption on u). From (7)

$$\begin{array}{rrcl} (i) & US - ps' & = & I \\ (ii) & Us - p\sigma & = & 0 \\ (iii) & -p'S & = & 0 \\ (iv) & -p's & = & 1 \end{array}$$

and (iii) means p is contained in the null space of S. Negativeness semi-definiteness follows from premultiplication of (i) by S to obtain $SUS = S$ from (iii), and $Sp = 0$ and the symmetry of S: so $\phi' S\phi \leq 0$ for any vector ϕ because $\phi' SUS\phi \leq 0$ from (3) in view of $p'S\phi = 0$. To see that $\text{rank}[S] = n - 1$, suppose that $S\psi = 0$ for a vector $\psi \neq 0$. Then $-ps'\psi = \psi$ by (i) so ψ and p are linearly dependent. Alternatively, if $\text{rank}[S] \leq n - 1$, then the first matrix on the left side of (6) must be singular, which contradicts the assumed non-singulariy of B defined in (4).

Appendix B.

(i) Proof of negative semidefiniteness of A.

From (16), $A = [I - px']U[I - xp']$, so

$$d'Ad = d'[I - px']U[I - xp']d.$$

Now consider the vector $[I - xp']d$: if this vector has the property that $p'[I - xp']d = 0$ for any d then we have the result we seek from the first order condition (1) and the utility maximisation condition (3). But $p'[I - xp'] = 0$ simply in view of $p'x = y = 1$.

(ii) Proof that rank$[A] = n - 1$, is both necessary and sufficient for the invertibility the differentials dp and dx for uncompensated and compensated inverse demands.

For necessity, recall from (13) that

$$dp = \mu\{[I - px']\,U - pu'_x\}dx$$

if A has rank less than $n-1$ then $[I - px']\,U$ has the same rank ($\leq n-2$), by the symmetry of U. Clearly $x'\,[I - px']\,U = 0$ and rank$[A] \leq n - 2$ means there is another vector, ϕ linearly independent of x such that $\phi' A = 0$ i.e., $\phi'[I - px']U = 0$. Now $x'p = 1$ in view of the normalisation and if $\phi' p = 0$ we have shown that the matrix in braces is singular; if $\phi' p \neq 0$, we can choose a linear combination of x and ϕ, ξ, say, such that $\xi' p = 0$ so that $\xi'\{[I - px']\,U - \mu p u'_x\} = 0$ so the matrix in braces is again singular. So the differentials of uncompensated inverse demands cannot be inverted if rank$[A] < n - 1$. The same is true of compensated inverse demands: from (15), $dp = \mu A dx$, whenever $p'dx = 0$; for invertibility of the differentials we require that there exists a matrix S, such that $Sdp = dx$ whenever $p'dx = 0$, i.e., $[SA - I]dx = 0$ for all such dx implying rank$[SA - I] \leq n - 1$, or rank$[SA] \geq n - 1$ and hence rank$[A] \geq n - 1$.

For sufficiency, suppose rank$[A] = n - 1$. We argue as at the end of section 3: From (17),

$$dp = [\mu A - bp']dx.$$

The matrix on the right hand side is nonsingular because rank$[A] = n-1$, since $x'b = 1$ (recalling $x'A = 0$) or equivalently $x'p = y \neq 0$. For compensated demands, again under the same assumptions as used in this section: from (15), $dp = \mu A dx$ for dx such that $p'dx = 0$, the constant utility condition. Now let $\mu = \lambda$: if S is a $n \times n$ matrix, then $Sdp/\mu = SAdx$ and we can choose S so that $SAdx = dx$ whenever $p'dx = 0$; this is possible because rank$[A] = n - 1$, and in view of this and $Ax = 0$, S can be chosen (not uniquely) so that $AS = I - xp'$. Then $Sdp/\mu = dx$ for such a choice of S.

Appendix C

Proof of rank$[A] = n - 1$ if and only if B is nonsingular.

Sufficiency. We show that if B is singular then rank$[A] \leq n - 2$. If B is singular, $Ud + pb = 0$, $p'd = 0$ for $d \neq 0$ and/or $b \neq 0$. but then $d'Ud + d'p = 0$, implying $d'U = 0$. But $d'A = d'[I - px']U[I - xp'] = d'U[I - px'] = 0$

because $d'p = 0$. Hence $d'A = 0$. Now if $d \neq 0$ we have a contradiction because $p'd = 0$ implies that d has both positive and negative elements, which means that d and x are linearly independent, and since $x'A = 0$ it follows that rank$[A] \leq n-2$. If $d = 0$ then $b = 0$, because of $Ud = 0$, contradicting the singularity of B.

Necessity. If rank$[A] \leq n-2$, then $\phi' A = 0$ for ϕ linearly independent of x and or $\phi'[I - px']U[I - xp'] = 0$, or $d'U = 0$ for $d' = [\phi' - (\phi' p)x']$. But if $Ud = 0$ then B is nonsingular only if $d'p \neq 0$; but this cannot be true since $d'p = \phi' p - (\phi' p)x'p = 0$ in view of $x'p = 1$.

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